Int. J. Solids Structures Vol. 20, No. 6, pp. 579-587, 1984
Printed in Great Britain.

0020-7683/84 $3.00 + .00

THERMAL STRESSES IN LAMINATED BEAMS

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(Received 24 January 1983; in revised form 18 August 1983)

Abstract—A displacement variation which incorporates the effects of transverse shear and normal strain is used to formulate the equations of equilibrium for a laminated beam subjected to external loads and temperature fields. The interlaminar stress field obtained from the present method is compared to those obtained from the theory of elasticity and the elementary beam theory. The numerical results indicate a substantial difference in the stress fields obtained from the present and the elementary beam theory for the cases of transverse and sinusoidal temperature fields. However, for uniform temperature distributions, the stresses are in close agreement.

1. INTRODUCTION

One of the first investigations dealing with thermally induced strains in bi-metallic beams was conducted by Timoshenko[1] in 1925. Elementary beam theory was used to obtain bending and axial stresses for beam sections sufficiently far away from the free end of the beam. More recent studies of the interlaminar stresses in laminated beams subjected to thermal and external loads are found in Refs.[2–6].

In general, most of the thermal analyses on laminated beams are based on the Bernoulli–Euler hypothesis which neglect the effect of transverse shear and normal deformations[7]. This approach is satisfactory for thin homogeneous beams. However, if the beam is thick or transversely heterogeneous, the transverse shear stresses may be large in comparison to the longitudinal stresses. Hence, the neglect of these transverse stresses and strains may lead to errors in the evaluation of the stress field in laminates.

The purpose of the present work is to determine the stress field of a laminated beam subjected to a temperature field. In order to describe the deformation field of each laminate accurately, the Bernoulli–Euler assumptions are relaxed by using displacement functions which consider the existence of the transverse strains[9, 10]. A system of $2(n-1)$ differential equations are formed by expressing the equilibrium equations in the longitudinal and transverse directions at the interface between the laminae. The system of equations are then solved for the $2(n-1)$ unknown transverse shear and normal stresses.

To illustrate the proposed procedure, the stress field of a simply supported beam subjected to a sinusoidal load is examined under constant, uniform, quadratic and sinusoidal temperature variations.

2. FORMULATION

In this section, the beam will be treated to have n laminae as indicated in Fig. 1. Each lamina, i, has thickness t_i, elastic constant E_i, Poissons ratio ν_i, and a coefficient of thermal expansion α_i.

2.1 *Displacement variation*

The displacement variation to be used within each lamina can be written as

$$u_{x_i} = u_0 - z_i D w_0 + \int_{\bar{z}=0}^{\bar{z}=z_i} e_{xz}\,\mathrm{d}\bar{z} - \int_{\bar{z}=0}^{\bar{z}=z_i} (z_i - \bar{z}) D e_{zz_i}\,\mathrm{d}\bar{z} \tag{1}$$

and

$$u_{z_i} = w_0 + \int_{\bar{z}=0}^{\bar{z}_i} e_{zz_i}\,\mathrm{d}z \tag{2}$$

where u_{x_i} and u_{z_i} are the displacements in the longitudinal and transverse directions

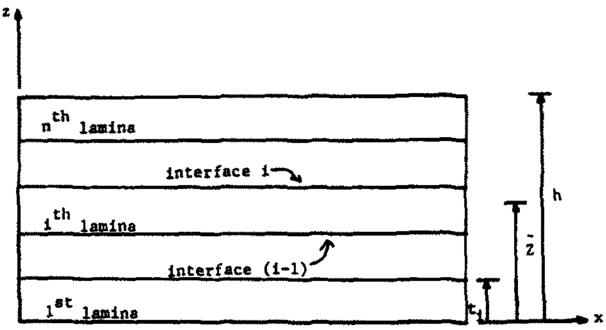

Fig. 1. Geometry of a laminated beam.

respectively. The integral terms represent the transverse normal and shear strain correction terms[9, 10]. Upon discretizing the liminated beam into n thin laminae of constant thickness t, and assuming that the strains within each thin lamina are linear, eqns (1) and (2) after further simplification are expressed in the summation form as follows

$$u_{x_i} = u_0 - z_i D w_0 + \sum_{j=1}^{i} t e_{xz_i} + e_{xz_i}(z_i - it) - \sum_{j=1}^{i} D e_{zz_j}\left[z_i t + \frac{t^2}{2}(1-2j)\right]$$

$$- D e_{zz_j}\frac{z_i^2}{2}(z_i it) - D e_{zz_i}\left(\frac{i^2 t^2}{2}\right) \tag{3}$$

and

$$u_{z_i} = w_0 + \sum_{j=1}^{j=i} t e_{zz_j} + e_{zz_i}(z_i - it) \tag{4}$$

where e_{zz_i} and e_{xz_i} are interpreted as the constant average value of the strain within the ith lamina.

2.2 *Displacement-strain relations and constitutive equations*

The corresponding strain functions for each lamina are obtained from Cauchy relations given as

$$e_{xx_i} = \frac{\partial u_{x_i}}{\partial x} \tag{5}$$

and

$$e_{zz_i} = \frac{\partial w_i}{\partial z}. \tag{6}$$

For the problem at hand the constitutive equations relating strain, stress, and temperature variation are expressed by generalized Hooke's law as

$$\sigma_{xx_i} = C_{11}^i(e_{xx_i} - \alpha_i T_i) + C_{13}^i(e_{zz_i} - \alpha_i T_i) \tag{7}$$

and

$$\sigma_{zz_i} = C_{13}^i(e_{xx_i} - \alpha_i T_i) + C_{33}^i(e_{zz_i} - \alpha_i T_i) \tag{8}$$

where C_{13}^i, C_{11}^i and C_{33}^i are plane stress material properties. Differentiating eqns (3) and (4) to obtain the strains within each lamina according to eqns (5) and (6) and substituting the strains into the constitutive relation, eqn (7), one obtains the thermoelastic relationship between stresses, neutral surface deformations, strains, temperature and material characteristics for the ith lamina. The longitudinal stress, σ_{xx_i}, then can be expressed as

$$
\begin{aligned}
\sigma_{xx_i} = {} & C_{11}^i Du_0 - C_{11}^i z_i D^2 w + C_{11}^i \sum_{j=1}^{i} tDe_{xz_j} + C_{11}^i De_{xz_i}(z_i - it) \\
& - C_{11}^i \sum_{j=1}^{i} D^2 e_{zz_j}\left[z_i t + \frac{t^2}{2}(1-2j)\right] - C_{11}^i D^2 e_{zz_i}\frac{z_i^2}{2} + C_{11}^i D^2 e_{zz_i}(z_i it) \\
& - C_{11}^i D^2 e_{zz_i}\left(\frac{i^2 t^2}{2}\right) + C_{13}^i e_{zz_i} - [C_{11} + C_{13}]\alpha_i T_i. \qquad (9)
\end{aligned}
$$

2.3 *Neutral surface deformations*

The force and bending moment resultants acting on the laminated beam can be found by integrating the longitudinal stress of eqn (9) through the total thickness of the laminate.

$$N = h\int_{\bar{z}=0}^{\bar{z}=z} \sigma_{xx_i}\,\mathrm{d}\bar{z} = h\sum_{i=1}^{n}\int_{(i-1)t}^{it} \sigma_{xx_i}\,\mathrm{d}z \qquad (10)$$

and

$$T = h\int_{\bar{z}=0}^{\bar{z}=z} \sigma_{xx_i}\bar{z}\,\mathrm{d}\bar{z} = h\sum_{i=1}^{n}\int_{(i-1)t}^{it} \sigma_{xx_i}\,\mathrm{d}z. \qquad (11)$$

Since the resultant can be evaluated for a given loading, eqns (10) and (11) can be rewritten to express the derivatives of the neutral surface deformations ($\mathrm{d}^2u/\mathrm{d}x^2$) and ($\mathrm{d}^3w/\mathrm{d}x^3$), as functions of the force and moments resultants and the unknown transverse shear and normal strain derivatives. This in return enables one to express eqn (9) only as a function of unknown transverse strains.

2.4 *Continuity of stresses*

For a thin lamina, the transverse stresses within each lamina may be expressed as an average value of the adjacent interlaminar stresses as defined in Fig. 2.

From the constitutive equations, one can relate the strain in the ith lamina to the adjacent interlaminar stresses. Hence, the normal and transverse shear strains can be

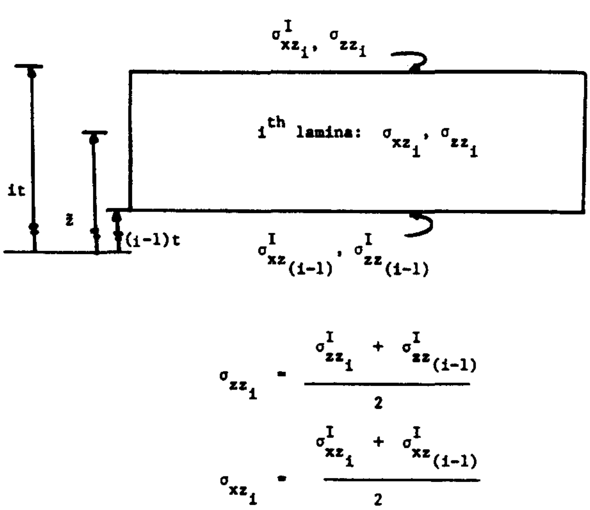

Fig. 2. Definition of interlaminar stresses in the ith lamina.

described as follows

$$e_{zz_i} = \frac{\sigma^I_{zz_i} + \sigma^I_{zz(i-1)}}{2C^i_{33}} - \frac{C^i_{13}}{C^i_{33}} e_{xx_i} \tag{12}$$

and

$$e_{xz_i} = \frac{\sigma^I_{xz_i} + \sigma^I_{xz(i-1)}}{2C^i_{33}}. \tag{13}$$

Since each lamina is assumed to be thin, the longitudinal strain e_{xx_i} in eqn (12) may be expressed as a function of the normal force and moment resultants as given by the classical beam theory. Since this substitution is made at the lamina level and only in eqn (12), its effect should not hinder the over all accuracy of the proposed development.

2.5 *Equations of equilibrium*

The equations of equilibrium in the longitudinal and transverse directions of a typical beam section of Fig. 3 are as follows

$$\int_{\bar{z}=0}^{\bar{z}=z} \frac{\partial \sigma_{xx_i}}{\partial x}\, dz - \sigma^I_{xz_L} + \sigma^I_{xz_i} = 0 \tag{14}$$

and

$$-\sigma_{zz_L} + \sigma^I_{zz_i} + \int_{\bar{z}=0}^{\bar{z}=z} \frac{\partial \sigma_{xz_i}}{\partial x}\, dz = 0. \tag{15}$$

The substitution of eqns (9)–(13) into eqns (14) and (15) yields the matrix representation of the equilibrium equations as follows

$$[FI] \begin{Bmatrix} D^3\sigma_{zz_1} \\ \vdots \\ D\sigma^I_{zz(n-1)} \end{Bmatrix} + [GI] \begin{Bmatrix} D\sigma^I_{zz_1} \\ \vdots \\ D\sigma^i_{zz(n-1)} \end{Bmatrix} + [HI] \begin{Bmatrix} D^2\sigma^I_{xz_1} \\ \vdots \\ D^2\sigma^I_{xz(n-1)} \end{Bmatrix}$$

$$+ [VIC11] \begin{Bmatrix} \sigma^I_{xz_1} \\ \vdots \\ \sigma^I_{xz(n-1)} \end{Bmatrix} + \{VIC\}D^3N + \{VIC1\}DN + \{VIC\}D^3T + \{VIC3\}DT$$

$$+ \{VIC4\}Tm_i + \{VIC6\}D^3\sigma^I_{zz_T} + \{VIC5\}D\sigma^I_{zz_L} + \{VIC7\}D\sigma^I_{zz_T} + \{VIC8\}D\sigma^I_{zz_L} \tag{16}$$

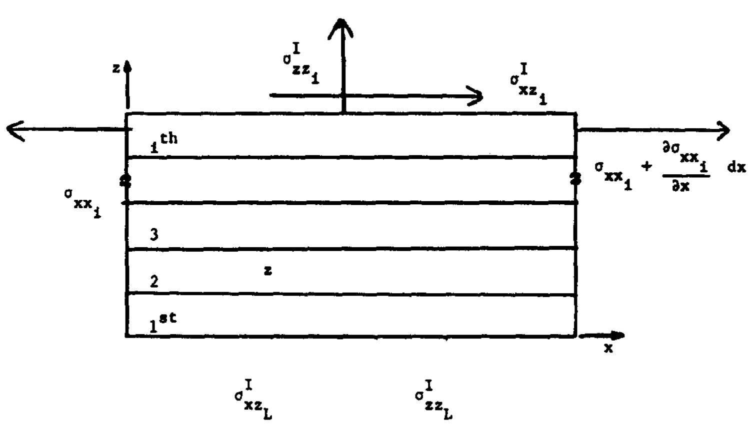

Fig. 3. Transverse normal and shear stresses acting on a section of the laminated beam.

and

$$\begin{Bmatrix} \sigma^{l}_{zz_1} \\ \vdots \\ \sigma^{l}_{zz_{(n-1)}} \end{Bmatrix} + [HH] \begin{Bmatrix} D\sigma^{l}_{xz_1} \\ \vdots \\ D\sigma^{l}_{xz_{(n-1)}} \end{Bmatrix} = \{VIC14\}\sigma^{l}_{zz_L} + \{VIC13\}D\sigma^{l}_{xz_L}. \tag{17}$$

2.6 *Solution procedure*

Equations (16) and (17) represent a system of $2(n - 1)$ non-homogeneous differential equations with coefficients given by geometrical and material characteristics. The method of undetermined coefficients is used to obtain the particular solution to the system of differential equations given by eqns (16) and (17). At this point the homogeneous solution is not addressed. In Refs. [9, 10], the homogeneous solution is interpreted as the transient solution, therefore it is assumed at this point that the particular solution is of greater importance.

3. NUMERICAL RESULTS

The example presented herein is the simply-supported heterogeneous beam of Fig. 4. It consists of eight laminae of equal thickness. The material characteristics of the laminae are given in Tables 1 and 2.

The moment, shear and upper surface tractions are given by

$$M = \frac{100\,L^2}{(m\pi)^2}\sin(m\pi x/L) \tag{18}$$

and

$$V = \frac{100\,L}{m\pi}\cos(m\pi x/L) \tag{19}$$

and

$$\sigma_{zz_L} = -100\sin m\pi x/L \tag{20}$$

where m is 4 and L is 24 in.

The temperature distributions that will be used are as follows

$$T = 120 \tag{21}$$

and

$$T = T_{m_i}(x) \tag{22}$$

and

$$T = T_{m_i}\sin(\pi x/L) \tag{23}$$

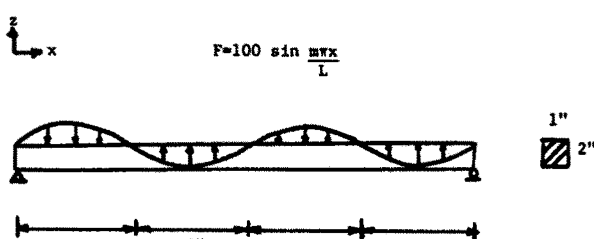

Fig. 4. Simply-supported beam with a sinusoidal load.

Table 1. Material layup for the laminated beams

Lamina	Material
1	Steel
2	Steel
3	Aluminum
4	Glass
5	Glass
6	Aluminum
7	Steel
8	Steel

Table 2. Material properties of the laminated beam

Property	Units	Steel	Glass	Aluminum
Poisson's Ratio		.3	.25	.27
Young's Modulus	psi	30×10^{6}	7.2×10^{6}	10×10^{6}
Shear Modulus	psi	10×10^{6}	3.8×10^{6}	3.7×10^{6}
Coefficient of Thermal Expansion	$1/^\circ F$	60×10^{-7}	5.4×10^{-6}	13.3×10^{-6}

and

$$T = T_{m_i}(2z/H + 1)^2(x/L)^2. \tag{24}$$

The interlaminar stresses may now be expressed as

$$\sigma^{I}_{zz_i} = \beta_i \sin \pi x/6 \tag{25}$$

and

$$\sigma^{I}_{xz_i} = \alpha_i \cos \pi x/6. \tag{26}$$

Upon substituting eqns (18)–(24) and their respective derivatives into eqns (16) and (17), one obtains a system of $2(n-1)$ differential equations with the coefficients β_i and α_i. Please note that "n" stands for the total number of laminae and which in this case is equal to eight.

$$-[FI]\begin{Bmatrix}\beta_1\\ \vdots\\ \beta_{n-1}\end{Bmatrix}\frac{n^3\pi^3}{L^3}+[GI]\begin{Bmatrix}\beta_1\\ \vdots\\ \beta_{n-1}\end{Bmatrix}\frac{n\pi}{L}-[HI]\begin{Bmatrix}\alpha_1\\ \vdots\\ \alpha_{n-1}\end{Bmatrix}\frac{n^2\pi^2}{L^2}$$

$$+[VIC11]\begin{Bmatrix}\alpha_1\\ \vdots\\ \alpha_{n-1}\end{Bmatrix}=\{VIC\}\frac{100\,\pi h}{L}+\{VIC1\}\frac{100\,Lh}{\pi}-\{VIC2\}\frac{100\,\pi n}{L}$$

$$+\{VIC3\}\frac{100\,L}{n\pi}+\{VIC4\}Tm_i-\{VIC6\}\frac{100\,\pi^3 n^3}{L^3}-\{VIC\}\frac{100\,\pi}{L} \tag{27}$$

and

$$\begin{Bmatrix} \beta_1 \\ \vdots \\ \beta_{n-1} \end{Bmatrix} - [HH] \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{Bmatrix} \frac{n\pi}{L} = 0. \tag{28}$$

Equation (28) is first solved for the vector $\{\beta\}$ and the resulting expression is substituted into eqn (27). Then the elements of the vector $\{\alpha\}$ in eqn (27) are found by solving the final equation by the Gaussian elimination technique. The matrix representation of eqns (27) and (28) lends itself conveniently for modular programming. This is accomplished by assigning different subroutines to form the matrices that are dependent on geometry and loading criteria. Tables 3–6 show the stress field in the laminated beam as obtained from the present formulation and the elementary beam theory.

Table 3. Stress field for a simply-supported laminated beam under mechanical and thermal fields, $T = 120°F$, $x = (L/2)$

Lamina	Material	Classical Solution - Non-Deformable Normals Stress psi			Present Solution - Normal and Shear Strain Correction Stress psi					
		σ_{xz_i}	σ_{zz_i}	σ_{xx_i}	σ_{xz_i}	% Dif.	σ_{zz_i}	% Dif.	σ_{xx_i}	% Dif.
1	Steel	47.58	5.00	-540.68	51.33	-8.5	5.82	- 6.4	-556.32	- 2.8
2	Steel	108.14	20.34	-443.30	111.75	- 3.3	22.12	- 8.7	-432.10	+ 2.5
3	Aluminum	132.12	36.61	-136.34	140.25	-6.1	38.75	- 5.8	-120.13	+11.8
4	Glass	156.87	48.65	- 68.36	163.67	-4.3	51.97	- 6.	- 74.03	- 8.8
5	Glass	155.49	63,33	68.36	164.31	-5.6	69.00	- 8.	74.34	- 6.9
6	Aluminum	132.53	78.36	136.35	141.80	-6.9	84.17	- 7.	120.20	+11.8
7	Steel	109.12	86.11	443.30	112.00	-2.6	96.43	- 5.	432.10	+ 2.5
8	Steel	46.50	88.00	540.65	51.33	-8.5	98.34	- 6.	556.32	- 2.8
Load: Sinusoidal										

Table 4. Stress field for a simply-supported laminated beam under mechanical and thermal fields, $T = T_{m_i}(x)$, $x = (L/2)$

Lamina	Material	Classical Solution - Non-Deformable Normals Stress psi			Present Solution - Normal and Shear Strain Correction Stress psi					
		σ_{xz_i}	σ_{zz_i}	σ_{xx_i}	σ_{xz_i}	% Dif.	σ_{zz_i}	% Dif.	σ_{xx_i}	% Dif.
1	Steel	33.97	0.35	-387.00	35.17	-3.4	0.38	- 8.7	-405.28	-4.5
2	Steel	85.43	0.50	-273.00	88.66	-3.6	0.55	-10.0	-287.23	-4.7
3	Aluminum	129.15	0.72	-106.00	136.34	-5.1	0.80	-11.1	-112.13	-5.4
4	Glass	143.00	6.86	- 45.63	142.00	+ .7	7.25	- 5.7	- 47.09	-4.4
5	Glass	142.95	10.36	45.66	142.15	0.0	11.16	- 7.7	47.00	-4.4
6	Aluminum	129.15	20.02	106.15	136.35	-5.1	22.15	-10.6	112.13	-5.3
7	Steel	85.44	28.11	273.76	88.00	-3.6	29.17	- 3.4	287.20	-4.9
8	Steel	33.92	33.14	387.00	35.17	-3.5	36.13	- 9.0	405.28	-4.5
Load: Sinusoidal										

Table 5. Stress field for a simply-supported laminated beam under mechanical and thermal fields, $T = Tm_t(2z/H + 1)^2 (x/L)^2$, $x = (L/2)$

Lamina	Material	Classical Solution - Non-Deformable Normals Stress psi			Present Solution - Normal and Shear Strain Correction Stress psi					
		σ_{xz_i}	σ_{zz_i}	σ_{xx_i}	σ_{xz_i}	% Dif.	σ_{zz_i}	% Dif.	σ_{xx_i}	% Dif.
1	Steel	23.34	0.05	- 119.44	27.32	-17.0	0.06	-20.	-122.66	-2.6
2	Steel	63.66	0.07	- 92.33	73.85	-16.0	0.08	-14.	- 97.01	-5.1
3	Aluminum	78.44	0.11	- 72.11	98.50	-25.6	0.13	-18.1	- 76.31	-5.5
4	Glass	93.10	0.93	- 52.13	112.9	-21.0	1.27	-36.6	- 54.00	-3.7
5	Glass	94.00	1.48	52.91	113.0	-20.62	1.37	-26.3	54.33	-3.7
6	Aluminum	78.31	3.64	72.32	99.16	-20.21	3.96	- 9.	76.24	-5.1
7	Steel	63.33	4.85	92.00	74.68	-13.2	6.60	-36.	97.01	-5.1
8	Steel	23.04	6.02	118.03	29.15	-26.5	7.11	-18.	121.95	-3.2
Load: Sinusoidal										

Table 6. Stress field for a simply-supported laminated beam under mechanical and thermal fields, $T = T_{m_t} \sin x/L$, $x = (L/2)$

Lamina	Material	Classical Solution - Non-Deformable Normals Stress psi			Present Solution - Normal and Shear Strain Correction Stress psi					
		σ_{xz_i}	σ_{zz_i}	σ_{xx_i}	σ_{xz_i}	% Dif.	σ_{zz_i}	% Dif.	σ_{xx_i}	% Dif.
1	Steel	38.58	3.00	- 413.21	40.02	-.3.5	3.85	-28.	-431.05	-4.3
2	Steel	80.32	7.25	- 257.61	91.03	-13.7	8.00	-10.	-272.93	-5.6
3	Aluminum	125.43	12.35	- 88.32	140.24	-11.8	15.10	-22.	- 95.21	-7.9
4	Glass	138.16	29.69	- 25.64	159.49	-15.4	35.15	-18.	- 27.33	-8.0
5	Glass	139.03	34.81	25.32	158.47	-13.9	38.10	- 9.0	27.33	-0.0
6	Aluminum	126.32	40.00	88.36	139.26	-10.2	46.71	-16.7	95.21	-7.9
7	Steel	80.36	47.33	257.61	92.23	-14.	52.13	-10.14	271.96	-5.6
8	Steel	37.90	54.23	413.21	41.26	- 8.8	63.00	-16.17	431.93	-4.2
Load: Sinusoidal										

4. DISCUSSION

The stress field of simply-supported, sinusoidally loaded laminated beam subjected to a constant temperature and a linear temperature distribution as indicated in Tables 3 and 4 do not show severe disagreements from the stresses obtained by the classical solution. The percentage differences between the stresses are within 10%. However, for a temperature variation which is a quadratic function of both longitudinal and transverse coordinates as displayed in Table 5, the transverse shear stresses differ by 17–26.5% through the thickness of a beam at a given x-coordinate location. The percentage differences for the transverse normal stress vary between 18 and 36.6%, whereas the longitudinal stress that may at times be used for design criteria remains relatively unaffected with a percentage difference of 2.6–5.1%. Similar observations can be made from the displayed data of Table 6 where a sinusoidal temperature field is used. The percentage differences are somewhat less than those tabulated in Table 5 but larger than those of Tables 3 and 4.

5. CONCLUSIONS

A displacement variation which incorporates the effects of transverse shear and normal strains is used to formulate a thermal stress analysis procedure for laminated beams. The interlaminar transverse shear and normal stresses in a heterogeneous beam are evaluated. It is observed that for the case of heterogeneous beams subjected to a sinusoidal or a transverse temperature distribution, the transverse normal and shear stresses deviate significantly from those obtained from classical beam theory. However, for uniform and linear temperature fields, the stresses obtained by the two different approaches were in close agreement.

REFERENCES

1. S. Timoshenko, Analysis of bi-metal thermostats. *J. Opt. Soc. Am.* **11**, 233–253 (1925).
2. P. B. Grimado, Interlaminar thermoelastic stresses in layered beams. *J. Thermal Stresses* **1**, 75–86 (1978).
3. T. S. Chow, Thermal warping of layered composities. *J. Appl. Phys.* **47**, 1351–1354 (1976).
4. B. A. Boley and J. H. Weiner, *Theory of Thermal Stresses*. Wiley, New York (1969).
5. C. S. Yang, Prevention of thermal bending of multi-layered beams and plates. *J. Appl. Phys.* **49**, 418–423 (1975).
6. D. Burgreen, *Elements of Thermal Stress Analysis*, 1st Edn (Edited by C. P. Press). Jamaica, New York (1971).
7. D. J. Johns, *Thermal Stress Analysis*. Pergamon Press, Oxford (1965).
8. T. J. Kozik, Linearly exact transverse displacement variations. *Am. Inst. Aeronaut. and Astronaut. J.* **8**, 366–370 (1970).
9. O. Ochoa, Transverse shear and normal stresses in laminated beams. Ph.D. Dissertation, Texas A & M University, College Station, Texas (1982).
10. V. M. Marcano, *Thermal stresses in laminated beams*. Masters Thesis, Texas A&M University, College Station, Texas (1982).